using monte carlo integration to solve the rendering equation

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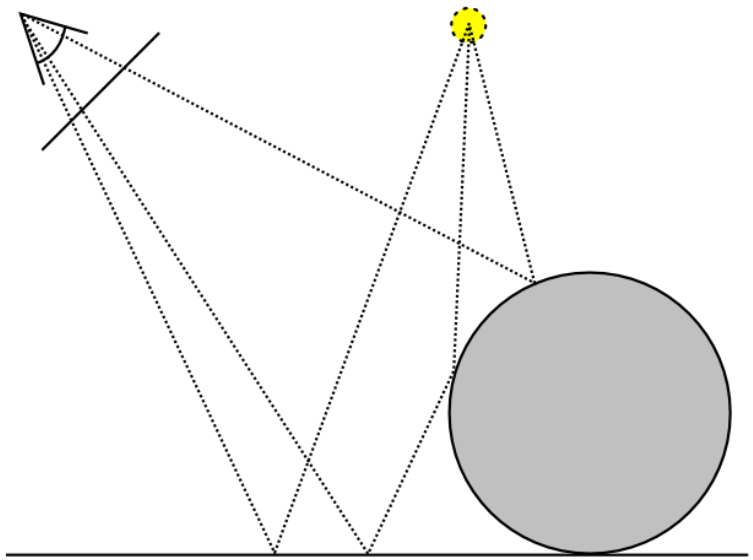
the rendering equation

computer-generated photo-realistic images are created by accurately simulating the way light interacts with the world

good simulations require good modeling of lighting, materials, lenses, and the transport of light



(not photo-real example)

the rendering equation

rendering systems, such as a ray tracer or path tracer, use the rendering equation to describe how the light bounces around the virtual scene until it enters the camera's lens

$$L_o(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\Omega} \rho(\mathbf{x}, \omega_i, \omega_o) L_i(\mathbf{x}, \omega_i)(\omega_i \cdot \mathbf{n}) d\omega_i$$

the rendering equation describes the total amount of light (L_o) leaving from a point ($\mathbf{x}$) along a particular direction (ω_o) given a function for all incoming light (L_i) about the hemisphere (Ω) and a reflectance function (ρ)

the rendering equation

$$L_o(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\Omega} \rho(\mathbf{x}, \omega_i, \omega_o) L_i(\mathbf{x}, \omega_i)(\omega_i \cdot \mathbf{n}) d\omega_i$$

L_e and ρ are relatively easy to define well, and the equation is fairly straight-forward, yet the results from such a simple equation can be quite stunning

[luc bégin, link]

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solving the rendering equation

so, how do we (efficiently) solve this?

$$L_o(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\Omega} \rho(\mathbf{x}, \omega_i, \omega_o) L_i(\mathbf{x}, \omega_i)(\omega_i \cdot \mathbf{n}) d\omega_i$$

notes:

- Ω is 2D domain (hemisphere)
- the equation is recursively defined (L_o, L_i)
- the function L_i is not well-behaved in general
  - discontinuous

solving the rendering equation

so, how do we (efficiently) solve this?

$$L_o(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_{\Omega} \rho(\mathbf{x}, \omega_i, \omega_o) L_i(\mathbf{x}, \omega_i)(\omega_i \cdot \mathbf{n}) d\omega_i$$

monte carlo integration to the rescue!

note

in the interest of time, the focus of this talk is to provide intuition and motivation, not necessarily for a deeper understanding of the application of statistics or in computer graphics

just sit back and enjoy

integrals and averages

integral of a function over a domain

$$\int_{\mathbf{x}\in D} f(\mathbf{x})dA_{\mathbf{x}}$$

"size" of a domain

$$A_D = \int_{\mathbf{x}\in D} dA_{\mathbf{x}}$$

average of a function over a domain

$$\frac{\int_{\mathbf{x}\in D} f(\mathbf{x})dA_{\mathbf{x}}}{\int_{\mathbf{x}\in D} dA_{\mathbf{x}}} = \frac{\int_{\mathbf{x}\in D} f(\mathbf{x})dA_{\mathbf{x}}}{A_D}$$

integrals and averages examples

average "daily" snowfall in Hillsboro last year

- domain: year, time interval (1D)
- integration variable: "day" of the year
- function: snowfall of "day"

$$\frac{\int_{day \in year} s(day) d length(day)}{length(year)}$$

integrals and averages examples

"today" average snowfall in Kansas

- domain: Kansas, surface (2D)
- integration variable: "location" in Kansas
- function: snowfall of "location"

$$\frac{\int_{location \in Kansas} s(location) darea(location)}{area(location)}$$

integrals and averages examples

"average" snowfall in Kansas per day this year

- domain: Kansas × year, area × time (3D)
- integration variables: "location" and "day" in Kansas this year
- function: snowfall of "location" and "day"

$$\frac{\int_{day \in year} \int_{loc \in KS} s(loc, day) d area(loc) d length(day)}{area(loc) length(day)}$$

discrete random variable

- random variable: x
- values: $x_0, x_1, ..., x_n$
- probabilities: $p_0, p_1, ..., p_n$, where $\sum_{j=1}^{n} p_j = 1$
- example: rolling a die
  - values: $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$
  - probabilities: $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

expected value and variance

- expected value: $E[x] = \sum_{j=1}^{n} v_j p_j$
  - average value of the variable
- variance: $\sigma^2[x] = E[(x - E[x])^2] = E[x^2] + E[x]^2$
  - how much different from the average
- example: rolling a die
  - expected value: $E[x] = (1 + 2 + 3 + 4 + 5 + 6)/6 = 3.5$
  - variance: $\sigma^2[x] = \ldots = 2.917$

estimating expected values

- to estimate the expected value of a variable
  - choose a set of random *values* based on the probability
  - average their results

$$E[x] \approx \frac{1}{N} \sum_{i=1}^{N} x_i$$

- larger N give better estimate
- example: rolling a die
  - roll 3 times: $\{3, 1, 6\} \rightarrow E[x] \approx (3 + 1 + 6)/3 = 3.33$
  - roll 9 times: $\{3, 1, 6, 2, 5, 3, 4, 6, 2\} \rightarrow E[x] \approx 3.51$

(strong) law of large numbers

- by taking *infinitely* many samples, the error between the estimate and the expected value is *statistically* zero
  - the estimate will converge to the right value

$$\mathrm{P}\left[E[x] = \lim_{N \to \mathrm{inf}} \frac{1}{N} \sum_{i=1}^{N} x_i\right] = 1$$

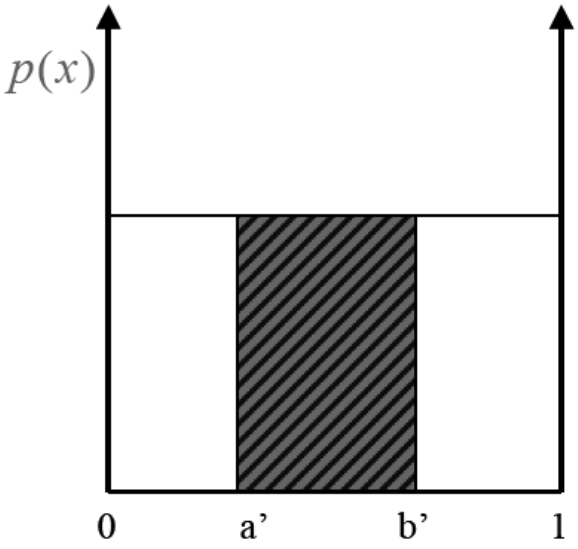
continuous random variable

- random variable: x
- values: $x \in [a, b]$
- probability density function: $x \sim p$
  - property: $\int_a^b p(x)dx = 1$
- probability that variable has value x: $p(x)$

uniformly distributed random variable

- p is the same everywhere in the interval
  - $p(x) = \text{const}$ and $\int_a^b p(x)dx = 1$ implies

$$p(x) = \frac{1}{b-a}$$



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expected value and variance

- expected value: $E[x] = \int_a^b xp(x)dx$
  - $E[g(x)] = \int_a^b g(x)p(x)dx$
- variance: $\sigma^2[x] = \int_a^b (x - E[x])^2 p(x)dx$
  - $\sigma^2[g(x)] = \int_a^b (g(x) - E[g(x)])^2 p(x)dx$
- estimating expected values: $E[g(x)] \approx \frac{1}{N} \sum_{i=1}^{N} g(x_i)$

multidimensional random variables

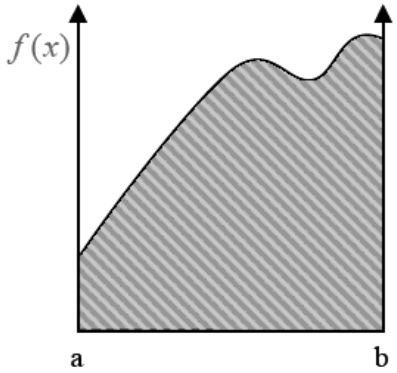
- everything works fine in multiple dimensions
  - but it is often hard to precisely define domain
    - except in simple cases

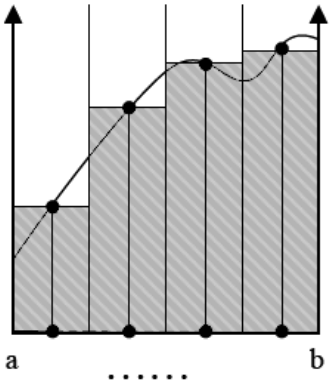
$$E[g(\mathbf{x})] = \int_{\mathbf{x} \in D} g(\mathbf{x}) p(\mathbf{x}) dA_{\mathbf{x}}$$

deterministic numerical integration

- split domain in set of fixed segments
- sum function values times size of segments

$$I = \int_a^b f(x)dx \qquad\qquad I \approx \sum_j f(x_j)\Delta x$$





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monte carlo numerical integration

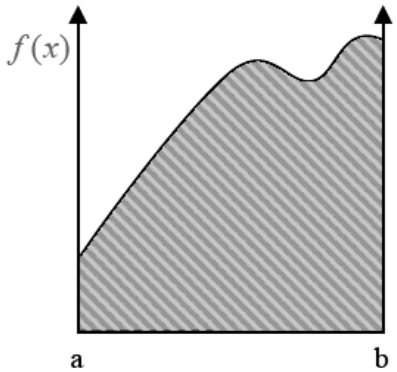
- need to evaluate: $I = \int_a^b f(x)dx$
- by definition: $E[g(x)] = \int_a^b g(x)p(x)dx$
- can be estimated as: $E[g(x)] \approx \frac{1}{N}\sum_{i=1}^{N} g(x_i)$
- by substitution: $g(x) = f(x)/p(x)$

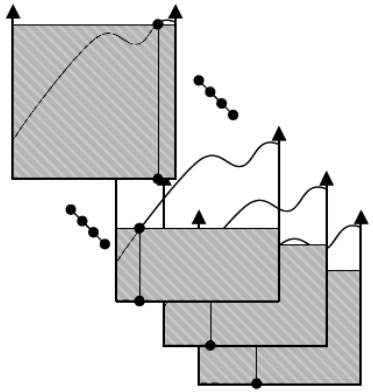
$$I = \int_a^b \frac{f(x)}{p(x)} p(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

monte carlo numerical integration

intuition: compute the area randomly and average the results

$$I = \int_a^b f(x)dx \qquad\qquad I \approx \bar{I} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}$$





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monte carlo numerical integration

formally, we can prove that

$$\bar{I} = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \qquad \Rightarrow \qquad E[\bar{I}] = E[g(x)]$$

meaning that if we were to try multiple times to evaluate the integral using our new procedure, we would get, on average, the same result

variance of the estimate: $\sigma^2[\bar{I}] = \frac{1}{N}\,\sigma^2[g(x)]$

example: integral of constant function

analytic integration

$$I = \int_a^b f(x)dx = \int_a^b kdx = k(b-a)$$

monte carlo integration

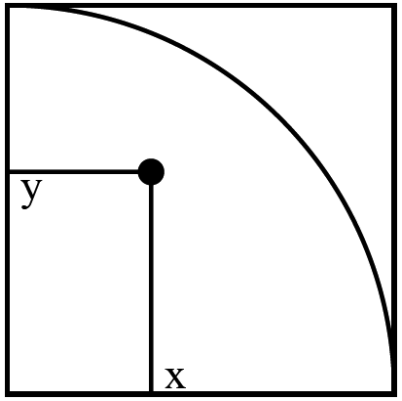
$$\begin{aligned} I &= \int_a^b f(x)dx = \int_a^b kdx \approx \\ &\approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)} = \frac{1}{N}\sum_{i=1}^{N}k(b-a) = \\ &= \frac{N}{N}k(b-a) = k(b-a) \end{aligned}$$

example: computing π

take the square $[0, 1]^2$ with a quarter-circle in it

$$A_{qcircle} = \int_0^1 \int_0^1 f(x, y) dx dx$$

$$f(x, y) = \begin{cases} 1 & (x, y) \in qcircle \\ 0 & \text{otherwise} \end{cases}$$



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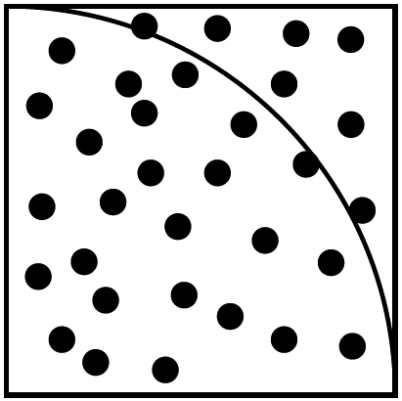
example: computing π

estimate area of quarter-circle by tossing point in the plane and evaluating f

$$A_{qcircle} \approx \frac{1}{N} \sum_{i=1}^{N} f(x_i, y_i)$$



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example: computing π

- by definition: $A_{qcircle} = \pi/4$
- numerical estimation of π
  - without any trig functions

$$\pi \approx \frac{4}{N} \sum_{i=1}^{N} f(x_i, y_i)$$

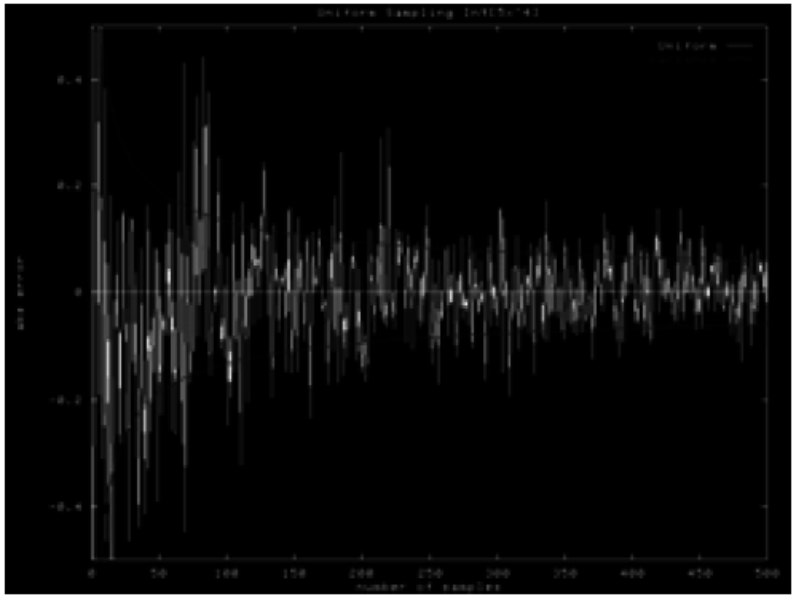
monte carlo numerical integration

- works in any dimension!
  - need to carefully pick the points
  - need to properly define the pdf
    - hard for complex domain shapes
    - e.g., how to uniformly sample a sphere?
- works for badly-behaving functions!

$$I = \int_{\mathbf{x} \in D} f(\mathbf{x}) dA_{\mathbf{x}} \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(\mathbf{x})}{p(\mathbf{x})}$$

monte carlo numerical integration

- expected value of the error is $O(1/\sqrt{N})$
  - convergence does not depend on dimensionality
  - deterministic integration is hard in high dimensions



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