Colorability and Determinants of $T(m, n, r, s)$ Twisted Torus Knots for $n \equiv \pm 1 \pmod{m}$*

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Abstract

We develop theorems for calculating the p-colorability of the families of $T(m, n, r, s)$ twisted torus knots for $n \equiv \pm 1 \pmod{m}$ by finding their determinants. Instead of the usual method of reducing crossing matrices to find the determinant, we describe a new method that is applicable for braid representations with full cycles and twists.

1 Introduction

Recently, Breiland, et al. [2] used determinants to completely characterize the p-colorability of torus knots. Conceptually, twisted torus knots, a recent addition to the field first described by Dean [5], are derived from torus knots. Thus, studying the determinants and p-colorability of twisted torus knots is a natural extension of Brieland, et al.

In our paper, we develop theorems for calculating the determinant of certain families of twisted torus knots $T(m, n, r, s)$, namely, when $n \equiv \pm 1 \pmod{m}$. Table 1 presents a summary of our results. The columns for m, r, and s give the parity of those parameters (if the column for s is left blank, that means the parity of s has no effect on the formula for the determinant). The column for n gives its congruence mod m, and the final column gives the determinant.

The organization of the paper is as follows. Section 2 provides background information and previously known results. Section 3 introduces a new method of finding the determinant of twisted torus knots and proves some preliminary results. In Section 4 we prove our main results. Finally, in Section 5, we conclude with suggestions for further research.

2 Background

2.1 Torus knots and twisted torus knots

For m, n relatively prime, let $T(m, n)$ represent the torus knot that circles the meridian of a torus m times and the longitude of a torus n times [1]. $T(m, n)$ is the closure of the braid with m strands and n cycles, where we define a *cycle* on m strands as the passing of the right-most strand over the remaining $m - 1$ strands.

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m	$n =$	r	s	$\det(T(m,n,r,s))$
even	$mq \pm 1$	even		$\lvert mq \pm 1 + rs \pm (m-r)\, qrs \rvert$
even	$mq \pm 1$	odd	odd	$\lvert r \pm (mr - r^2 + 1)\, q \rvert$
even	$mq \pm 1$	odd	even	$\lvert mq \pm 1 \rvert$
odd	$2mq \pm 1$	even		$\lvert rs \pm 1 \rvert$
odd	$2mq \pm 1$	odd	odd	r
odd	$2mq \pm 1$	odd	even	1
odd	$(2q+1)m \pm 1$	even		$\lvert m \mp (m-r)\, rs \rvert$
odd	$(2q+1)m \pm 1$	odd	odd	$\lvert mr - r^2 + 1 \rvert$
odd	$(2q+1)m \pm 1$	odd	even	m

Table 1: Summary of determinants of $T(m,n,r,s)$ twisted torus knots with $n \equiv \pm 1 \pmod m$

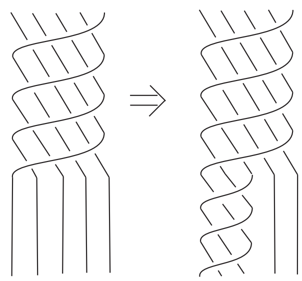

Figure 1: The $T(5,4)$ torus knot changed into a $T(5,4,3,1)$ twisted torus knot

A *twisted torus knot* can be constructed by beginning with the braid representation of a $T(m,n)$ torus knot and then performing s full twists on r parallel strands [4]. We denote a twisted torus knot by $T(m,n,r,s)$, where m is the total number of strands in the braid representation, n is the number of cycles on the m strands, r is the number of strands to be twisted, and s is the number of full twists on the r strands, as in Figure 1. Obviously, m and r must be positive and $r \leq m$. Both n and s can be positive or negative, hence there are four possibilities for the signs of the parameters. However, the determinant and p-colorability are the same for a knot and its mirror image, so we assume that n is positive throughout.

An important equivalence that we will use several times is described in the following theorem, which was shown by Dean [5] for $s = \pm 1$. His arguments can be extended to any value for s.

Theorem 2.1. *The $T(m,n,r,s)$ twisted torus knot is equivalent to the $T(n,m,r,s)$ twisted torus knot.*

2.2 Colorability and determinants

A knot is *p-colorable* if the strands in a projection of the knot can be labelled according to the following three conditions [6]. The first is that each strand must be labeled with an integer from 0 to $p-1$. The second requires that at least two labels are distinct. The third requires that

$$x + y - 2z \equiv 0 \pmod p \tag{1}$$

at each crossing, where z is the label of the overstrand and x and y are the labels of the two understrands [6]. Note that if a knot is colorable for some prime p, then it is colorable for any multiple of p.

A knot is p-colorable if and only if p divides the determinant of the knot. The *determinant* of a knot is the absolute value of the determinant of a minor crossing matrix constructed by removing

m	n	r	s	$det(C)$	m	n	r	s	$det(C)$
4	**3**	**2**	**1**	**1**	**5**	**3**	**3**	**1**	**-3**
4	3	2	2	-1	5	3	3	2	-1
4	3	2	3	-3	5	3	3	3	-3
4	3	2	4	-5	5	3	3	4	-1
4	3	2	5	-7	5	3	3	5	-3
4	**3**	**3**	**1**	**1**	**5**	**3**	**4**	**1**	**-1**
4	3	3	2	3	5	3	4	2	-1
4	3	3	3	1	5	3	4	3	-1
4	3	3	4	3	5	3	4	4	-1
4	3	3	5	1	5	3	4	5	-1
5	**3**	**2**	**1**	**1**	**5**	**4**	**2**	**1**	**11**
5	3	2	2	3	5	4	2	2	17
5	3	2	3	5	5	4	2	3	23
5	3	2	4	7	5	4	2	4	29
5	3	2	5	9	5	4	2	5	35

Table 2: Experimental data on the determinants of twisted torus knot minor crossing matrices

a row and a column from the crossing matrix of a projection of the knot. A crossing matrix is a matrix representing the system of equations determined by requirement (1) at each crossing of a projection of the knot [6].

The following result of Breiland, et al. completely characterizes the colorability of torus knots [2]. Recall that $T(m,n)$ and $T(n,m)$ are the same knot, so only two cases need to be considered.

Theorem 2.2. *Let $T(m,n)$ be a torus knot and p a prime,*
i) if m and n are both odd, then $T(m,n)$ is not p-colorable.
ii) if m is odd and n is even, then $T(m,n)$ is p-colorable if and only if $p \mid m$.

Their proof was a direct consequence of the following lemma, which they proved by evaluating Alexandar polynomials at $t = -1$ [6].

Lemma 2.3. *For any torus knot $T(m,n)$,*
i) if m and n are odd, then $det(T(m,n)) = 1$.
ii) if m is odd and n is even, then $det(T(m,n)) = m$.

3 Methods

3.1 Computer Experimentation

We wrote a program in MATLAB that input the four parameters of a twisted torus knot and output the determinant of a minor crossing matrix of the knot, which is equal to the determinant of the knot up to sign. Table 2 is a sample of the program's output. The boldface lines identify the beginning of a new "family," where we fix m, n, and r, and let s vary.

When r is even, the determinants of the $T(m,n,r,s)$ twisted torus knots form an arithmetic progression in s. When r is odd, the determinants oscillate between two values as s varies. Two questions naturally arise–what determines the starting values and differences in the progressions and what determines the values in the oscillations? In trying to answer these questions, we were able to make conjectures for several families of twisted torus knots. The next two subsections develop the techniques that we used to prove our conjectures.

3.2 Definitions and notation

We define a *coloring vector* as a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ that lists the colors of m strands of a twisted torus knot from right to left between two consecutive cycles (for example, see the top of Figure 2). We also define a *coloring matrix* as a matrix that operates on a coloring vector according to the coloring relation (1). A coloring matrix represents the changes that occur to the colors on the m strands after a specified number of cycles and/or twists.

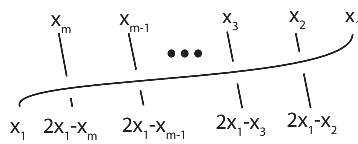

Figure 2: One cycle of an arbitrary knot

We define Γ_m to be the coloring matrix that represents the change after one cycle of m strands. Therefore, for a twisted torus knot with m strands and n cycles, the coloring matrix that represents the changes through the torus part (the part above the twists) of the knot is Γ_m^n. The Γ_m matrix representing one cycle of an arbitrary knot is an $m \times m$ matrix of the form

$$\Gamma_m = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ 2 & 0 & -1 & \ldots & 0 & 0 \\ 2 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \qquad (2)$$

as can be seen from Figure 2 (also see Breiland, et al. [2]).

We define χ_r as a coloring matrix that represents the change that occurs after one full twist of r strands in the lower part of a twisted torus knot projection. By definition, $\chi_r = \Gamma_r^r$, since there will be r cycles on r strands in one full twist. Later in this section we will explore special properties of some powers of χ_r matrices. Some of these properties have previously been stated by Przytycki, using n-moves and half-twists [7].

Throughout, we will use χ_r to symbolize the $r \times r$ matrix that represents the changes occuring on only the r strands that are being twisted, and also to symbolize the $m \times m$ matrix that represents the changes on all m strands in the lower part of the diagram. In this case, the rightmost $m - r$ strands are left unchanged, so this matrix will contain the original χ_r matrix in the lower right, while also having 1's in the main diagonal from the upper left corner down to the start of the original χ_r matrix. We hope that the distinction will be clear from the context.

If $A_1, A_2, \ldots, A_i$ are coloring matrices that represent all of the changes that occur to the coloring vectors, in order, from the top of a projection of a twisted torus knot to the bottom, then we can form an overall coloring matrix for the twisted torus knot $A = A_i A_{i-1} \ldots A_1$. Then, if $\mathbf{x}$ is the coloring vector at the top of the projection, the coloring vector $\mathbf{x}'$ at the bottom of the projection can be found using $A\mathbf{x} = \mathbf{x}'$. Thus, the twisted torus knot can be colored if and only if there exists a nonconstant vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{x}$. In our calculations, A is generally equal to $\chi_r^s \Gamma_m^n$ for the twisted torus knot $T(m, n, r, s)$. For an example, see Figure 3.

3.3 Determinants

The usual method of assessing p-colorability of a knot depends on the fact that the system of equations obtained from the coloring relation (1) at each crossing has a nontrivial solution mod p

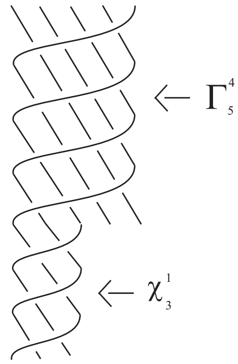

Figure 3: Coloring matrices for the $T(5,4,3,1)$ twisted torus knot

if and only if any minor of the crossing matrix of the knot has determinant divisible by p [6]. Here we describe a different method for finding the determinant of a twisted torus knot that utilizes coloring matrices rather than crossing matrices. This method has the advantage of dealing with much smaller matrices, which have some very nice forms and useful properties.

Recall that a knot has a nontrivial coloring if and only if there is a nonconstant vector $\mathbf{x}$ such that $\mathbf{x} = A\mathbf{x} \bmod p$ for the coloring matrix A. So, we analyze the system of equations $B\mathbf{x} = \mathbf{0} \bmod p$, where $B = A - I$. This has nontrivial solutions; namely, any nonzero multiple of the constant vector $\mathbf{x} = (1, 1, \ldots, 1)$. Also, we are free to set some $x_i = 0$. Thus, in looking for nonconstant solutions, we can delete a row and column from B, forming a minor that we denote as B'. Now, the knot has a nontrivial p-coloring if and only if p divides the determinant of B'. Therefore, the absolute value of the determinant of B' is exactly the determinant of the knot.

3.4 Forms of matrices

Recall that the coloring matrix χ_k corresponds to a full twist on k strands. The form of χ_k is

$$\begin{pmatrix} 1 & -2 & 2 & \ldots & 2 & -2 & 2 \\ 2 & -3 & 2 & \ldots & 2 & -2 & 2 \\ 2 & -2 & 1 & \ldots & 2 & -2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & -2 & 2 & \ldots & 1 & -2 & 2 \\ 2 & -2 & 2 & \ldots & 2 & -3 & 2 \\ 2 & -2 & 2 & \ldots & 2 & -2 & 1 \end{pmatrix} \tag{3}$$

when k is odd, and

$$\begin{pmatrix} 3 & -2 & 2 & \ldots & 2 & -2 & 2 & -2 \\ 2 & -1 & 2 & \ldots & 2 & -2 & 2 & -2 \\ 2 & -2 & 3 & \ldots & 2 & -2 & 2 & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & -2 & 2 & \ldots & 3 & -2 & 2 & -2 \\ 2 & -2 & 2 & \ldots & 2 & -1 & 2 & -2 \\ 2 & -2 & 2 & \ldots & 2 & -2 & 3 & -2 \\ 2 & -2 & 2 & \ldots & 2 & -2 & 2 & -1 \end{pmatrix} \tag{4}$$

when k is even, as can be shown by induction.

3.5 Properties of coloring matrices

Let χ_k be a coloring matrix, with k odd. Then, χ_k has the form (3). Squaring this immediately yields the following lemma. Its corrolary is similar to a result of Przytycki [7].

Lemma 3.1. *For k odd, we have $\chi_k^2 = I_k$.*

Corollary 3.2. *An even twist of an odd number of strands applied to a p-colorable torus knot or twisted torus knot will result in a new knot that is also p-colorable.*

Proof. Since $\chi_k^2 = I_k$ for k odd, it follows that any even twist of an odd number of strands will have the same colors at the top and bottom. □

One can prove by induction that the coloring matrix χ_k^q, for k even, will have the form

$$\begin{pmatrix} 2q+1 & -2q & 2q & \dots & 2q & -2q \\ 2q & -2q+1 & 2q & \dots & 2q & -2q \\ 2q & -2q & 2q+1 & \dots & 2q & -2q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2q & -2q & 2q & \dots & 2q+1 & -2q \\ 2q & -2q & 2q & \dots & 2q & -2q+1 \end{pmatrix}. \tag{5}$$

Given this result, we can immediately prove another lemma. Again, a result similar to its corrolary was also demonstrated by Przytycki [7].

Lemma 3.3. *For k even, we have $\chi_k^q \equiv I_k$ mod q.*

Obviously, we could have stated that for k even, $\chi_k^q \equiv I_k$ mod $2q$. However, in this paper, we will only utilize the result as given in the lemma.

Corollary 3.4. *If the original torus knot was p-colorable, twisting an even number of strands s times, where $p|s$, will result in another p-colorable knot.*

Proof. We have $\chi_k^s = \chi_k^{pj}$ for some j. Then, $\chi_k^{pj} = I_k^j = I_k \pmod{p}$. Therefore, when coloring mod p, the same colors will appear at the top and bottom of the twist. □

In our proofs, we will use a few special powers of the Γ_m matrices, which we now calculate. First, we find Γ_m^{mq+1} for m even. This is equal to $\Gamma_m^{mq}\Gamma_m = \chi_m^q\Gamma_m$. This is (5) times (2), which is

$$\begin{pmatrix} 2q+2 & -2q-1 & 2q & -2q & \dots & -2q & 2q & -2q \\ 2q+2 & -2q & 2q-1 & -2q & \dots & -2q & 2q & -2q \\ 2q+2 & -2q & 2q & -2q-1 & \dots & -2q & 2q & -2q \\ 2q+2 & -2q & 2q & -2q & \dots & -2q & 2q & -2q \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2q+2 & -2q & 2q & -2q & \dots & -2q & 2q-1 & -2q \\ 2q+2 & -2q & 2q & -2q & \dots & -2q & 2q & -2q-1 \\ 2q+1 & -2q & 2q & -2q & \dots & -2q & 2q & -2q \end{pmatrix}. \tag{6}$$

Here, we exhibit the form of Γ_m^{mq-1} for m even, which is

$$\begin{pmatrix} 2q & -2q & 2q & \dots & 2q & -2q+1 \\ 2q-1 & -2q & 2q & \dots & 2q & -2q+2 \\ 2q & -2q-1 & 2q & \dots & 2q & -2q+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2q & -2q & 2q & \dots & 2q & -2q+2 \\ 2q & -2q & 2q & \dots & 2q-1 & -2q+2 \end{pmatrix} \tag{7}$$

When we multiply (7) by (2), we obtain (5). Therefore, the matrix (7) has been shown to be Γ_m^{mq-1}, since we have $\Gamma_m^{mq-1}\Gamma_m = \Gamma_m^{mq} = \chi_m^q$.

Finally, we calculate $\Gamma_m^{2mq\pm 1}$ for m odd. Since $\chi_m^{2q} = I_m$,

$$\Gamma_m^{2mq+1} = \Gamma_m^{2mq}\Gamma_m = I_m\Gamma_m = \Gamma_m. \tag{8}$$

Also,

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 2 \\ 0 & -1 & 0 & \dots & 0 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 2 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \tag{9}$$

times (2) is equal to I_m. Thus the matrix (9) is equal to Γ_m^{2mq-1}, since $\Gamma_m^{2mq-1}\Gamma_m = \Gamma_m^{2mq} = I_m$.

4 Results

In this section, we calculate the determinants of $T(m,n,r,s)$, for some families of the parameters. First we find $A = \chi_r^s\Gamma_m^n$, and then we use the process outlined in Section 3.3 for finding the determinant of the knot by finding the determinant of a minor of $A - I$, which we do by row reduction. We use the second definition of χ_r matrices given in Section 3.2–that is, a χ_r matrix is an $m \times m$ matrix that contains $m - r$ ones along the main diagonal, and then the rest of the nonzero entries in the lower right of the matrix. For r even, these will be

$$\chi_r^s = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 2s+1 & -2s & \dots & 2s & -2s \\ 0 & 0 & \dots & 0 & 2s & -2s+1 & \dots & 2s & -2s \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 2s & -2s & \dots & 2s+1 & -2s \\ 0 & 0 & \dots & 0 & 2s & -2s & \dots & 2s & -2s+1 \end{pmatrix}. \tag{10}$$

Recall from Lemma 3.1 that $\chi_r^2 = I_r$ for r odd. For r, s odd we have

$$\chi_r^s = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & -2 & \dots & -2 & 2 \\ 0 & 0 & \dots & 0 & 2 & -3 & \dots & -2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 2 & -2 & \dots & -3 & 2 \\ 0 & 0 & \dots & 0 & 2 & -2 & \dots & -2 & 1 \end{pmatrix}. \tag{11}$$

4.1 $T(m, mq+1, r, s)$ family with m even

By Theorem 2.1, the $T(4,5,2,s)$ family of twisted torus knots is the same as the $T(5,4,2,s)$ family of twisted torus knots. By Table 2, we see that this family has determinants in an arithmetic progression with starting value 5 (the determinant of $T(4,5)$) and difference 6. This is a special case of the following theorem, which states that related families of twisted torus knots will have determinants in arithmetic progressions with starting values at the determinant of the (untwisted) torus knot and a difference that depends on m, n, r, and s.

Theorem 4.1. *A $T(m, mq+1, r, s)$ twisted torus knot, with m, r even and $m > r$, has determinant $\Delta = |mq + 1 + rs + (m-r)\, qrs|$.*

Proof. Multiply the χ_r^s matrix on the right by Γ_m^{mq+1}. This is (10) multiplied by (6), yielding

$$\begin{pmatrix}
2q+2 & -2q-1 & \dots & -2q & 2q & -2q & 2q & \dots & 2q & -2q \\
2q+2 & -2q & \dots & -2q & 2q & -2q & 2q & \dots & 2q & -2q \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q+2 & -2q & \dots & -2q & 2q-1 & -2q & 2q & \dots & 2q & -2q \\
2q+2s+2 & -2q & \dots & -2q & 2q & -2q-2s-1 & 2q+2s & \dots & 2q+2s & -2q-2s \\
2q+2s+2 & -2q & \dots & -2q & 2q & -2q-2s & 2q+2s-1 & \dots & 2q+2s & -2q-2s \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q+2s+2 & -2q & \dots & -2q & 2q & -2q-2s & 2q+2s & \dots & 2q+2s & -2q-2s-1 \\
2q+2s+1 & -2q & \dots & -2q & 2q & -2q-2s & 2q+2s & \dots & 2q+2s & -2q-2s
\end{pmatrix}.$$

Here, R_{m-r+1} is the first row with entries that contain an s. We subtract I_m and remove the first row and column:

$$\begin{pmatrix}
-2q-1 & 2q-1 & \dots & -2q & 2q & -2q & 2q & \dots & 2q & -2q \\
-2q & 2q-1 & \dots & -2q & 2q & -2q & 2q & \dots & 2q & -2q \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-2q & 2q & \dots & -2q-1 & 2q-1 & -2q & 2q & \dots & 2q & -2q \\
-2q & 2q & \dots & -2q & 2q-1 & -2q-2s-1 & 2q+2s & \dots & 2q+2s & -2q-2s \\
-2q & 2q & \dots & -2q & 2q & -2q-2s-1 & 2q+2s-1 & \dots & 2q+2s & -2q-2s \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-2q & 2q & \dots & -2q & 2q & -2q-2s & 2q+2s & \dots & 2q+2s-1 & -2q-2s-1 \\
-2q & 2q & \dots & -2q & 2q & -2q-2s & 2q+2s & \dots & 2q+2s & -2q-2s-1
\end{pmatrix}.$$

To find the determinant of this matrix, we use elementary row operations to convert the matrix into an upper triangular matrix, whose determinant we can then easily compute by taking the product of the diagonal entries. Using the row operations $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3, \dots, R_{m-2} \to R_{m-2} - R_{m-1}$ yields the matrix

$$\begin{pmatrix}
-1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & -1 & 0 & 1+2s & -2s & \dots & 2s & -2s & 2s \\
0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 \\
-2q & 2q & -2q & 2q & \dots & -2q & 2q & -\alpha & \alpha & \dots & -\alpha & \alpha & -\alpha-1
\end{pmatrix},$$

where $\alpha = 2q+2s$. (Note that the $\pm 2s$'s occur in row R_{m-r-1}.) We now reduce the last row using

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{(m-r)/2} 2iq(R_{2i} - R_{2i-1}), \text{ and}$$

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{(r-2)/2} ((m-r)(1+2is)q + 2i(q+s))(R_{m-r+2i} - R_{m-r+2i-1}).$$

This leaves us with

$$\begin{pmatrix} -1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 1+2s & -2s & \dots & 2s & -2s & 2s \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \Delta \end{pmatrix},$$

where $\Delta = -1 - 2q - 2s - q\,(2s)\,(m-r) - ((m-r)\,(1+(r-2)\,s)\,q + (r-2)\,(q+s))$. The determinant of this upper triangular matrix is Δ, since there are an even number of -1's along the diagonal. We can rewrite Δ as $-1 - mq - rs - (m-r)\,qrs$. As we explained in Section 3.3, the determinant of the knot is the absolute value of the determinant of this matrix, so it follows that the determinant of the knot is equal to $|1 + mq + rs + (m-r)\,qrs|$. □

For these values of m and n but odd r, a different phenomenon results. For example, the $T\,(5,4,3,s)$ family has determinants that oscillate between 5 (the determinant of $T\,(5,4)$) and 7. Next we show that this is representative of related families of twisted torus knots, which have determinants that oscillate between the determinant of the untwisted knot and another value that depends on m, n, and r. We first prove the following lemma, for $s = 1$.

Lemma 4.2. *A $T\,(m, mq+1, r, 1)$ twisted torus knot, with m even and r odd, has determinant $\Delta = \left|r + \left(mr - r^2 + 1\right) q\right|$.*

Proof. Multiply the χ_r^s matrix by Γ_m^{mq+1}. This is (11) times (6), which equals

$$\begin{pmatrix} 2q+2 & -2q-1 & \dots & 2q & -2q & 2q & -2q & \dots & 2q & -2q \\ 2q+2 & -2q & \dots & 2q & -2q & 2q & -2q & \dots & 2q & -2q \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2q+2 & -2q & \dots & 2q & -2q-1 & 2q & -2q & \dots & 2q & -2q \\ 2q & -2q & \dots & 2q & -2q & 2q-1 & -2q+2 & \dots & 2q-2 & -2q+2 \\ 2q & -2q & \dots & 2q & -2q & 2q-2 & -2q+3 & \dots & 2q-2 & -2q+2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2q & -2q & \dots & 2q & -2q & 2q-2 & -2q+2 & \dots & 2q-1 & -2q+3 \\ 2q & -2q & \dots & 2q & -2q & 2q-2 & -2q+2 & \dots & 2q-2 & -2q+3 \\ 2q+1 & -2q & \dots & 2q & -2q & 2q-2 & -2q+2 & \dots & 2q-2 & -2q+2 \end{pmatrix}.$$

Note the change from row R_{m-r} to R_{m-r+1}. Subtract I_m and remove the first row and column:

$$\begin{pmatrix}
-2q-1 & 2q-1 & \dots & 2q & -2q & 2q & -2q & \dots & 2q & -2q \\
-2q & 2q-1 & \dots & 2q & -2q & 2q & -2q & \dots & 2q & -2q \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-2q & 2q & \dots & 2q-1 & -2q-1 & 2q & -2q & \dots & 2q & -2q \\
-2q & 2q & \dots & 2q & -2q-1 & 2q-1 & -2q+2 & \dots & 2q-2 & -2q+2 \\
-2q & 2q & \dots & 2q & -2q & 2q-3 & -2q+3 & \dots & 2q-2 & -2q+2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-2q & 2q & \dots & 2q & -2q & 2q-2 & -2q+2 & \dots & 2q-1 & -2q+2 \\
-2q & 2q & \dots & 2q & -2q & 2q-2 & -2q+2 & \dots & 2q-3 & -2q+3 \\
-2q & 2q & \dots & 2q & -2q & 2q-2 & -2q+2 & \dots & 2q-2 & -2q+1
\end{pmatrix}.$$

Reducing with $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3, \dots, R_{m-2} \to R_{m-2} - R_{m-1}$ gives

$$\begin{pmatrix}
-1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & -1 & 0 & 1 & -2 & \dots & -2 & 2 & -2 \\
0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \\
-2q & 2q & -2q & 2q & \dots & 2q & -2q & 2q-2 & -2q+2 & \dots & -2q+2 & 2q-2 & -2q+2
\end{pmatrix},$$

where the row containing the ± 2's and is R_{m-r-1}. We now reduce the last row using the operations

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{(m-r-1)/2} 2iq(R_{2i} - R_{2i-1}), \text{ and}$$

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{(r-3)/2} (((2i+1)(m-r)+1)q+2i)R_{m-r+2i} - \sum_{i=1}^{(r-1)/2} (((2i-1)(m-r)+1)q+2i)R_{m-r+2i-1}.$$

We now have the upper triangular matrix

$$\begin{pmatrix}
-1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & -1 & 0 & 1 & -2 & \dots & -2 & 2 & -2 \\
0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \Delta
\end{pmatrix},$$

where $\Delta = 1-2q-2(m-r-1)q+((r-2)(m-r)+1)q+r-3-2(((r-2)(m-r)+1)+r-1)$. Since there are an even number of -1's on the diagonal, the determinant is Δ, which simplifies to $-r-\left(mr-r^2+1\right)q$. The determinant of the knot is then $\left|r+\left(mr-r^2+1\right)q\right|$. □

This immediately leads into a theorem:

Theorem 4.3. *A $T(m, mq+1, r, s)$ twisted torus knot, with m even and r odd, has determinant $\Delta = \left|r+\left(mr-r^2+1\right)q\right|$ if s is odd, and determinant $\Delta = |mq+1|$ if s is even.*

Proof. If s is odd, χ_r^s will equal the one used in the proof of Lemma 4.2, so the determinant of $T(m, mq+1, r, s)$ would equal that of $T(m, mq+1, r, 1)$. If s is even, χ_r^s will be the identity, so the determinant of the knot would simply be the determinant of the $T(m, mq+1)$ torus knot, which is $mq+1$ by Lemma 2.3, since m is even and $mq+1$ is odd. □

4.2 $T(m, mq-1, r, s)$ family with m even

We now proceed to investigate a similiar family to the one just analyzed. In these proofs, instead of using some power of Γ_m that has a diagonal with -1's in it to the upper right of the main diagonal, as in (6), we utilize different powers of Γ_m that have the property that there is a diagonal with -1's in it to the lower left of the main diagonal, as in (7). By glancing at the values for the $T(4,3,2,s)$ family in Table 2, we conjecture that we will have an arithmetic progression beginning at the determinant of the $T(4,3)$ torus knot. We now prove that this is the case.

Theorem 4.4. *A $T(m, mq-1, r, s)$ twisted torus knot, with m, r even, has determinant $\Delta = |mq-1+rs-(m-r)qrs|$.*

Proof. Multiply the χ_r^s matrix by Γ_m^{mq-1}. This will be (10) times (7), which is

$$\begin{pmatrix}
2q & -2q & \dots & 2q & -2q & 2q & -2q & \dots & 2q & -2q+1 \\
2q-1 & -2q & \dots & 2q & -2q & 2q & -2q & \dots & 2q & -2q+2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q & -2q & \dots & 2q-1 & -2q & 2q & -2q & \dots & 2q & -2q+2 \\
2q & -2q & \dots & 2q & -2q-2s-1 & 2q+2s & -2q-2s & \dots & 2q+2s & -2q+2 \\
2q & -2q & \dots & 2q & -2q-2s & 2q+2s-1 & -2q-2s & \dots & 2q+2s & -2q+2 \\
2q & -2q & \dots & 2q & -2q-2s & 2q+2s & -2q-2s-1 & \dots & 2q+2s & -2q+2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q & -2q & \dots & 2q & -2q-2s & 2q+2s & -2q-2s & \dots & 2q+2s & -2q+2 \\
2q & -2q & \dots & 2q & -2q-2s & 2q+2s & -2q-2s & \dots & 2q+2s-1 & -2q+2
\end{pmatrix}.$$

We subtract I_m from this. At this point, instead of deleting the first row and column as we have done previously, we choose to remove the last row and column:

$$\begin{pmatrix}
2q-1 & -2q & \dots & 2q & -2q & 2q & \dots & -2q & 2q \\
2q-1 & -2q-1 & \dots & 2q & -2q & 2q & \dots & -2q & 2q \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q & -2q & \dots & 2q-1 & -2q-1 & 2q & \dots & -2q & 2q \\
2q & -2q & \dots & 2q & -2q-2s-1 & 2q+2s-1 & \dots & -2q-2s & 2q+2s \\
2q & -2q & \dots & 2q & -2q-2s & 2q+2s-1 & \dots & -2q-2s & 2q+2s \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q & -2q & \dots & 2q & -2q-2s & 2q+2s & \dots & -2q-2s-1 & 2q+2s \\
2q & -2q & \dots & 2q & -2q-2s & 2q+2s & \dots & -2q-2s-1 & 2q+2s-1
\end{pmatrix}.$$

The first row with entries containing a term with an s is R_{m-r+1}. We now reduce using the row operations $R_2 \to R_2 - R_3, R_3 \to R_3 - R_4, \ldots, R_{m-2} \to R_{m-2} - R_{m-1}, R_{m-1} \to R_{m-1} - R_1$ and $R_1 \to R_1 + R_{m-1}$. Additionally, we cyclically permute the rows by moving R_1 to the bottom, while shifting all of the other rows up by one. This puts the diagonal of -1's on the main diagonal using an even number of switches. Thus, the determinant remains unchanged. The matrix becomes

$$\begin{pmatrix}
-1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & -1 & 2s & -2s+1 & 2s & \ldots & -2s & 2s & -2s \\
0 & 0 & 0 & \ldots & 0 & -1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & -1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & -1 & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & -2s & 2s & -2s & \ldots & 2s & -2s-1 & 2s-1 \\
2q & -2q & 2q & \ldots & 2q & -\alpha & \alpha & -\alpha & \ldots & \alpha & -\alpha-1 & \alpha-1
\end{pmatrix},$$

where R_{m-r-1} is the first row with $\pm 2s$'s. (As before, $\alpha = 2q + 2s$.) We now reduce R_{m-2} with

$$R_{m-2} \to R_{m-2} + \sum_{i=1}^{(m-2)/2} R_{2i-1}.$$

We then reduce R_{m-1} with

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{(m-r-2)/2} 2iq(R_{2i-1} - R_{2i}) + (m-r)qR_{m-r-1},$$

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{r/2} ((m-r)(2iqs - q) - (2i-2)q - 2is)R_{m-r-2+2i}, \text{ and}$$

$$R_{m-1} \to R_{m-1} - \sum_{i=1}^{(r-2)/2} ((m-r)(2iqs - q) - 2iq - 2is)R_{m-r-1+2i}.$$

After this, we have successfully converted the matrix into an upper-triangular matrix with determinant $\Delta = 2q + 2s - 1 - 2s\,(m-r)\,q - ((m-r)\,((r-2)\,qs - q) - (r-2)\,q - (r-2)\,s)$:

$$\begin{pmatrix}
-1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & -1 & 2s & -2s+1 & 2s & \ldots & -2s & 2s & -2s \\
0 & 0 & 0 & \ldots & 0 & -1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & -1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & -1 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \Delta
\end{pmatrix}.$$

As before, there are an even number of -1's on the diagonal, and the row operations did not affect the determinant. Simplifying Δ, the determinant of the knot is $|-1 + mq + rs - (m - r)\, qrs|$. $\square$

To investigate this family when r is odd, we begin with a lemma for the case $s = 1$.

Lemma 4.5. *A $T(m, mq - 1, r, 1)$ twisted torus knot, with m even and r odd, has determinant $\Delta = \left|r - \left(mr - r^2 + 1\right) q\right|$.*

Proof. Multiply the χ_r^s matrix by Γ_m^{mq-1}. This is (11) multiplied by (7), which gives

$$\begin{pmatrix}
2q & -2q & \dots & -2q & 2q & -2q & 2q & \dots & -2q & 2q & -2q+1 \\
2q-1 & -2q & \dots & -2q & 2q & -2q & 2q & \dots & -2q & 2q & -2q+2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2q & -2q & \dots & -2q-1 & 2q & -2q & 2q & \dots & -2q & 2q & -2q+2 \\
2q & -2q & \dots & -2q & 2q-1 & -2q+2 & 2q-2 & \dots & -2q+2 & 2q-2 & -2q+2 \\
2q & -2q & \dots & -2q & 2q-2 & -2q+3 & 2q-2 & \dots & -2q+2 & 2q-2 & -2q+2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2q & -2q & \dots & -2q & 2q-2 & -2q+2 & 2q-2 & \dots & -2q+2 & 2q-2 & -2q+2 \\
2q & -2q & \dots & -2q & 2q-2 & -2q+2 & 2q-2 & \dots & -2q+3 & 2q-2 & -2q+2 \\
2q & -2q & \dots & -2q & 2q-2 & -2q+2 & 2q-2 & \dots & -2q+2 & 2q-1 & -2q+2
\end{pmatrix}.$$

As in the proof of Theorem 4.4, we delete the last row and column after subtracting I_m:

$$\begin{pmatrix}
2q-1 & -2q & \dots & -2q & 2q & -2q & 2q & \dots & -2q & 2q \\
2q-1 & -2q-1 & \dots & -2q & 2q & -2q & 2q & \dots & -2q & 2q \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q & -2q & \dots & -2q-1 & 2q-1 & -2q & 2q & \dots & -2q & 2q \\
2q & -2q & \dots & -2q & 2q-1 & -2q+1 & 2q-2 & \dots & -2q+2 & 2q-2 \\
2q & -2q & \dots & -2q & 2q-2 & -2q+3 & 2q-3 & \dots & -2q+2 & 2q-2 \\
2q & -2q & \dots & -2q & 2q-2 & -2q+2 & 2q-1 & \dots & -2q+2 & 2q-2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2q & -2q & \dots & -2q & 2q-2 & -2q+2 & 2q-2 & \dots & -2q+2 & 2q-2 \\
2q & -2q & \dots & -2q & 2q-2 & -2q+2 & 2q-2 & \dots & -2q+1 & 2q-2 \\
2q & -2q & \dots & -2q & 2q-2 & -2q+2 & 2q-2 & \dots & -2q+3 & 2q-3
\end{pmatrix}.$$

We apply $R_2 \to R_2 - R_3, R_3 \to R_3 - R_4, \dots, R_{m-2} \to R_{m-2} - R_{m-1}.R_{m-1} \to R_{m-1} - R_1$ and $R_1 \to R_1 + R_{m-1}$. Also, R_1 is moved to the bottom, and the other rows are shifted up one, giving

$$\begin{pmatrix}
-1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & -1 & 0 & -1 & 2 & -2 & \dots & -2 & 2 & -2 & 2 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -2 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\
-1 & 0 & 0 & 0 & \dots & 0 & -2 & 2 & -2 & 2 & \dots & 2 & -2 & 3 & -3 \\
2q & -2q & 2q & -2q & \dots & -2q & \beta & -\beta & \beta & -\beta & \dots & -\beta & \beta & -\beta+1 & \beta-1
\end{pmatrix}.$$

Here, R_{m-r-1} contains the sequence of alternating ± 2's and $\beta = 2q-2$. The absolute value of the determinant is unchanged by these row operations. To reduce R_{m-2}, we use

$$R_{m-2} \to R_{m-2} + \sum_{i=1}^{(m-2)/2} R_{2i-1}.$$

In so doing, we find that adding R_{m-r-1} to it creates a lot of cancellation. For the last row, we use

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{(m-r-1)/2} 2qi(R_{2i-1} - R_{2i}),$$

$$R_{m-1} \to R_{m-1} - \sum_{i=1}^{(r-1)/2} (((2i-1)(m-r)+1)q - 2i)R_{m-r-2+2i}, \text{ and}$$

$$R_{m-1} \to R_{m-1} - \sum_{i=1}^{(r-1)/2} (((2i+1)(m-r)-1)q - 2i)R_{m-r-1+2i}.$$

Our matrix has been transformed into

$$\begin{pmatrix}
-1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & -1 & 0 & -1 & 2 & -2 & \dots & -2 & 2 & -2 & 2 \\
0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 & 1 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -2 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \Delta
\end{pmatrix},$$

for $\Delta = 2q-3-2(m-r-1)q - (((r-2)(m-r)+1)q-(r-1)) + 2((r(m-r)-1)q-(r-2))$. There are $m-r-1$ entries of -1 on the main diagonal. Since $m-r-1$ is even, the determinant of this matrix is Δ, which simplifies to $-r+\left(mr-r^2+1\right)q$. The determinant of the knot is then $\left|r-\left(mr-r^2+1\right)q\right|$. □

As in the proof of Theorem 4.3, this lemma leads directly to a corresponding theorem.

Theorem 4.6. *A $T(m, mq-1, r, s)$ twisted torus knot, with m even and r odd, has determinant $\Delta = \left|r-\left(mr-r^2+1\right)q\right|$ if s is odd, and determinant $\Delta = |mq-1|$ if s is even.*

4.3 $T(m, 2mq+1, r, s)$ family with m odd

Now we begin our discussion of twisted torus knots when both m and n are odd. This represents a major change for two reasons. First, the $T(m,n)$ torus knot that we begin with will no longer be p-colorable for any p; by Lemma 2.3, it will have a determinant of 1. Additionally, the powers of the Γ_m matrices that we use will no longer have q's in them. However, after examination of Table 2, the trend of having either an oscillating pattern or an arithmetic progression appears to hold when

m and n are both odd (the determinants of the $T(5,3,4,s)$ family form an arithmetic progression with difference 0). Although the details are slightly different, the methods of this section closely follow those of Section 4.1. For space considerations, we suppress the matrices involved and only record the arithmetic details. We trust that the reader could supply the matrices if desired.

Theorem 4.7. *A $T(m, 2mq+1, r, s)$ twisted torus knot, with m odd, r even, and $m > r$, has determinant $\Delta = |rs+1|$.*

Proof. Multiply the χ_r^s matrix by Γ_m^{2mq+1}. By (8), this will be (10) times (2). As we did in Section 4.1, we will return to our method of subtracting I_m and removing the first row and column. We do not have to reduce any of the first $m-r$ rows, as there are no entries to the left of the long diagonal in these rows. (The first row containing 2's and -2's happens to be R_{m-r}.) Therefore, we use a different process of row operations, as we only will work with the last r rows, as follows: $R_{m-r+1} \to R_{m-r+1} - R_{m-r+2}, R_{m-r+2} \to R_{m-r+2} - R_{m-r+3}, \ldots, R_{m-2} \to R_{m-2} - R_{m-1}$.

All that remains is to reduce R_{m-1}. Our procedure for doing this is

$$R_{m-1} \to R_{m-1} + \sum_{i=1}^{(r-2)/2} 2si(R_{m-r+2i} - R_{m-r+2i-1}).$$

This converts the matrix into an upper triangular matrix with an odd number of -1's along the diagonal and $-\Delta = -2s - 1 - (r-2)s$ as the only other diagonal entry. The determinant of this matrix is then $\Delta = 1 + rs$. The determinant of the knot is thus $|1+rs|$. □

Similarly, we can prove that when r is odd the determinants will oscillate. However, they now oscillate between 1 and some other value, as the determinant of a $T(m, 2mq+1)$ torus knot is 1 by Lemma 2.3, because both m and $2mq+1$ are odd.

Lemma 4.8. *A $T(m, 2mq+1, r, 1)$ twisted torus knot, with m, r odd, has determinant $\Delta = r$.*

Proof. Multiply the χ_r^s matrix by Γ_m^{2mq+1}. By (8), we have (11) multiplied by (2). We subtract I_m and remove the first row and column. Again, we do not have to reduce the first $m-r$ rows. (The first row with more than two entries is R_{m-r}.) We use the following operations on the remaining rows: $R_{m-r+1} \to R_{m-r+1} - R_{m-r+2}, R_{m-r+2} \to R_{m-r+2} - R_{m-r+3}, \ldots, R_{m-2} \to R_{m-2} - R_{m-1}$.

The last row is the only one preventing an upper-triangular matrix. We remedy this with

$$R_{m-1} \to R_{m-1} - \sum_{i=1}^{(r-3)/2} 2i(R_{m-r+2i} + R_{m-r+2i-1}) - (r-1)R_{m-2}.$$

This leaves an upper triangular matrix with an odd number of -1's on the diagonal and $-\Delta$ in the last diagonal entry, where $-\Delta = 1 + (r-3) - 2(r-1)$. The determinant of this upper triangular matrix is Δ. Fortunately, Δ simplifies to r. The determinant of the knot is then just r. (Note that r can never be negative, as it represents the number of strands.) □

Again this lemma leads to a full theorem.

Theorem 4.9. *A $T(m, 2mq+1, r, s)$ twisted torus knot, with m, r odd, has determinant $\Delta = r$ if s is odd, and determinant $\Delta = 1$ if s is even.*

4.4 $T(m, 2mq-1, r, s)$ family with m odd

The final family that we will investigate with our procedure is the $T(m, 2mq-1, r, s)$ family. In many ways, these proofs correspond to those presented in Section 4.2, which deal with the $T(m, mq-1, r, s)$ family, just as the proofs from Section 4.3 correspond to those from Section 4.1. This is due to the fact that the diagonal with -1's is to the lower left of the main diagonal, instead of the upper right. As in the previous section we suppress the matrices to save space.

Theorem 4.10. *A $T(m, 2mq-1, r, s)$ twisted torus knot, with m odd, r even, and $m > r$, has determinant $\Delta = |rs-1|$.*

Proof. Multiply the χ_r^s matrix by Γ_m^{2mq-1}, which is (10) times (9). As in the proofs of Theorem 4.4 and Lemma 4.5, we opt to delete the last row and column after subtracting I_m. Here, the first row with $\pm 2s$'s R_{m-r+1}. In this proof, we use a different method of turning this matrix into a triangular matrix. Instead of subtracting each row from the row above it and ending up with an upper triangular matrix, we choose to subtract each row from the row below it, eventually reaching a lower triangular matrix. This avoids any need to cyclically permute the rows. Our row operations are $R_{m-1} \to R_{m-1} - R_{m-2}, R_{m-2} \to R_{m-2} - R_{m-3}, \ldots, R_{m-r+2} \to R_{m-r+2} - R_{m-r+1}$.

Because of our different procedure, we must reduce R_{m-r+1} (not R_{m-1}). We use

$$R_{m-r+1} \to R_{m-r+1} + \sum_{i=1}^{(r-2)/2} 2is(R_{m-2i+1} - R_{m-2i}).$$

This gives a lower triangular matrix with an odd number of $-1's$ along the diagonal and $-\Delta = 2s - 1 + (r-2)s$ in row R_{m-r+1} as the only other entry on the diagonal. The determinant of this matrix is $\Delta = -1 + rs$, and so the determinant of the knot is $|-1+rs|$. □

Our final proof of this type investigates a case where r is odd. Again, we are confirmed by Table 2, in which one family satisfying the following conditions is $T(5, 3, 3, s)$.

Lemma 4.11. *A $T(m, 2mq-1, r, 1)$ twisted torus knot, with m, r odd, and $m > r$, has determinant $\Delta = r$.*

Proof. Multiply the χ_r^s matrix by Γ_m^{2mq-1}. This will be (11) multiplied by (9). As in the proof of Theorem 4.10, we subtract I_m and remove the last row and column. We again choose to subtract each row (beginning with R_{m-r+1}) from the row below it, with the intention of finding a lower-triangular matrix. Our row operations are $R_{m-1} \to R_{m-1} - R_{m-2}, R_{m-2} \to R_{m-2} - R_{m-3}, \ldots, R_{m-r+2} \to R_{m-r+2} - R_{m-r+1}$.

All that remains is to reduce R_{m-r+1}, which we do with

$$R_{m-r+1} \to R_{m-r+1} - \sum_{i=1}^{(r-1)/2} 2iR_{m-2i+1} - \sum_{i=1}^{(r-3)/2} 2iR_{m-2i}.$$

This leaves a lower triangular matrix with an odd number of -1's along the diagonal, with the only other entry on the diagonal being $-\Delta = 1 + (r-3) - 2(r-1)$ in R_{m-r+1}. The determinant of this matrix is $\Delta = r$. Thus, the determinant of the knot is r (which is always positive). □

Naturally, this lemma gives a similar theorem.

Theorem 4.12. *A $T(m, 2mq-1, r, s)$ twisted torus knot, with m, r odd, has determinant $\Delta = r$ if s is odd, and determinant $\Delta = 1$ if s is even.*

4.5 $T(m, (2q+1)m+1, r, s)$ and $T(m, (2q+1)m-1, r, s)$ families with m odd

In this section, we use our previous results to prove some important corollaries.

Corollary 4.13. *The determinant of a $T(m, (2q+1)m+1, r, s)$ twisted torus knot is $\Delta = |mr - r^2 + 1|$ for m, r, s odd, and $\Delta = m$ for m, r odd and s even.*

Proof. First, consider the case of $T(m, m+1, r, s)$. Using Theorem 2.1, we rewrite this knot as $T(m+1, m, r, s)$. By Theorem 4.6, we see that its determinant is $\Delta = \left|r - \left((m+1)r - r^2 + 1\right)\right| = |mr - r^2 + 1|$ for s odd, and $\Delta = m$ for s even. Therefore, these are the determinants for the $T(m, m+1, r, s)$ knots. Since $\chi_m^2 = I_m$ by Lemma 3.1, adding $2qm$ cycles doesn't change the determinant, so $\det(T(m, (2q+1)m+1, r, s)) = \det(T(m+1, m, r, s))$ for any q. □

The following three corollaries similarly follow from Theorems 4.4, 4.3, and 4.1, respectively.

Corollary 4.14. *The determinant of a $T(m,(2q+1)m+1,r,s)$ twisted torus knot is* $\Delta = |m-(m-r)rs|$ *for m odd and r even.*

Corollary 4.15. *The determinant of a $T(m,(2q+1)m-1,r,s)$ twisted torus knot is* $\Delta = \left|mr-r^2+1\right|$ *for m,r,s odd, and $\Delta = m$ for m,r odd and s even.*

Corollary 4.16. *The determinant of a $T(m,(2q+1)m-1,r,s)$ twisted torus knot is* $\Delta = |m+(m-r)rs|$ *for m odd and r even.*

These four corollaries, together with the theorems presented in Sections 4.3 and 4.4, complete all cases when $n \equiv \pm 1 \pmod m$. This is because if $n \equiv \pm 1 \pmod m$, then $n \equiv \pm 1 \pmod{2m}$ or $n \equiv \pm m+1 \pmod{2m}$. The theorems from Sections 4.3 and 4.4 took care of $n \equiv \pm 1 \pmod{2m}$, while the four corollaries here fully covered the cases $n \equiv \pm m+1 \pmod{2m}$.

4.6 Counting p-colorings

The *p-nullity* of a knot is the dimension of the mod p nullspace of a crossing matrix for the knot. A knot with p-nullity n has $p^n - p$ different p-colorings, because there are n strands that can be assigned any of p different colors, whereas the remaining strands are then determined (subtracting p discards the trivial "colorings") [3]. Two colorings of a knot are *fundamentally different* if they are not simply permutations of each other. If two colorings are fundamentally different, then they belong to different *p-coloring classes*; otherwise, they are in the same p-coloring class. Brieland, et al. showed that if a torus knot is p-colorable, then it has only one nontrivial p-coloring class [2]. Our methods show a similar result for the twisted torus knots that we analyzed.

Theorem 4.17. *If a twisted torus knot $T(m,n,r,s)$, with $n \equiv \pm 1 \pmod m$, is p-colorable, then it has $p^2 - p$ different p-colorings, and hence only 1 nontrivial p-coloring class.*

Proof. In each of our proofs, B' was converted into a triangular matrix by row reduction. Note that all of the row operations were valid mod p for any p, and so the mod p nullspace of the matrix was unchanged. After reduction, all but one of the entries on the main diagonal were equal to ± 1. If the knot being analyzed was p-colorable–that is, if $p|\Delta$–then there was only one value on the diagonal of the reduced matrix that was divisible by p Thus, in assigning the values of the labels to the top strands, there were two free variables: one for the deleted column, and one for the column containing $\pm\Delta$. This implies that the p-nullity of the knot was 2. □

5 Conclusion

While the theorems presented in this paper provide examples of determinants from each of the possible combinations of the parities of the parameters of twisted torus knots, they do not completely characterize the determinants of all twisted torus knots. A natural goal would be a complete characterization. It may be possible to generalize the methods presented in this paper to all twisted torus knots; however, the families investigated in this paper were chosen because their matrices allowed for straightforward row-reduction schemes.

Future research could also investigate the patterns in labellings of twisted torus knots, two examples of which are shown in Figure 6. Breiland et al. [2] showed that all possible p-colorings of a torus knot were equivalent under permutation of the labels to a "main coloring," which arose from labeling the uppermost strands of their projection with $0, 1, \ldots, p-1$, in that order. However, many p-colorable twisted torus knots cannot be colored in this fashion–for example, the $T(4,5,2,1)$ twisted torus knot, which has determinant 11 by Theorem 4.1, cannot be 11-colored this way. Alternatively, the $T(5,4,3,2)$ twisted torus knot, which has determinant 5 by Corollary 4.15, can be 5-colored using the main coloring. It would be interesting to determine which twisted torus knots can be p-colored using the main coloring.

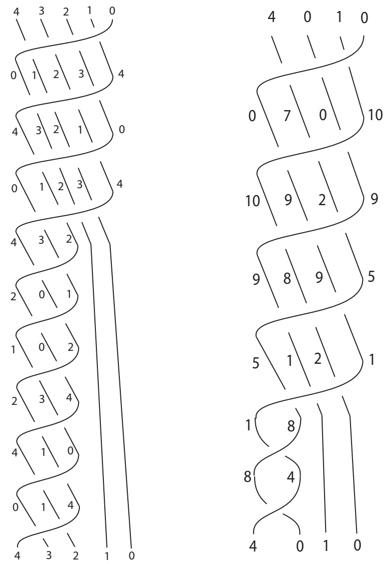

Figure 4: A 5-coloring of the $T(5,4,3,2)$ twisted torus knot and an 11-coloring of the $T(4,5,2,1)$ twisted torus knot

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References

[1] C. Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society, Providence, RI (2004).

[2] A. Breiland, L. Oesper, and L. Taalman, *p-Coloring Classes of Torus Knots*, James Madison University (2004), pre-print at `http://www.math.jmu.edu/~taal/`

[3] K. Brownell, K. O'Neil, and L. Taalman, *Counting m-Coloring classes of knots and links*, Pi Mu Epsilon Journal, Vol. 12, No. 5 (Fall 2006) 265-278.

[4] A. Champanerkar, I. Kofman, and E. Patterson, *The Next Simplest Hyperbolic Knots*, Journal of Knot Theory and Its Ramifications, Vol. 13, No. 7 (2004) 965-987.

[5] J. Dean, *Hyperbolic Knots With Small Seifert-Fibered Dehn Surgeries*, Ph.D. thesis, University of Texas at Austin (1996), available at `http://citeseer.ist.psu.edu/dean96hyperbolic.html`

[6] C. Livingston, *Knot Theory*, The Mathematical Association of America, Washington, DC (1993).

[7] J. Przytycki, *3-Coloring and Other Elementary Invariants of Knots*, pre-print at `http://arXiv.org/abs/math.GT/0608172`.